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SINGULAR SEMI-LINEAR EQUATIONS IN L1(IR).(U)

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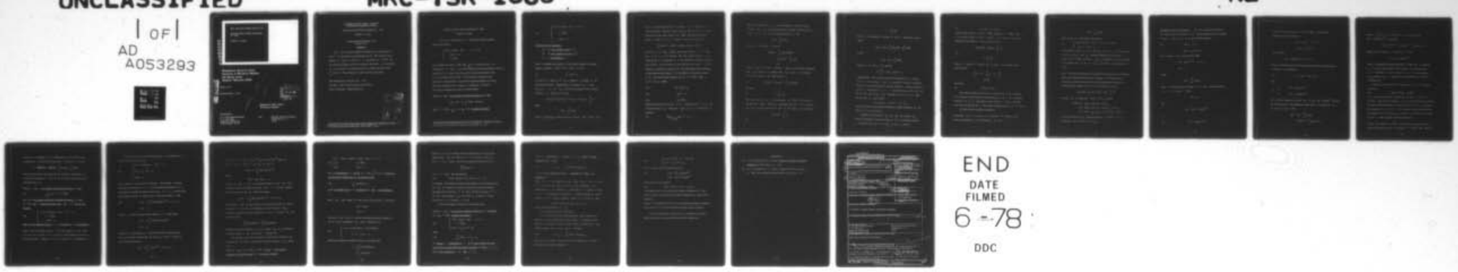

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SINGULAR SEMI-LINEAR EQUATIONS IN $L^1(\mathbb{R})$

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ABSTRACT

Let g be a positive continuous function on $\mathbb{R}$ which tends to zero at $-\infty$ and which is not integrable over $\mathbb{R}$. The boundary-value problem $-u'' + g(u) = f$, $u'(\pm\infty) = 0$, is considered for $f \in L^1(\mathbb{R})$. We show that this problem can have a solution if and only if g is integrable at $-\infty$ and if this is so then the problem is solvable precisely when $\int_{-\infty}^{\infty} f(t)dt > 0$. Some extensions of this result are also given.

AMS (MOS) Subject Classification - 34B15

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Work Unit Number 1 (Applied Analysis)

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SINGULAR SEMI-LINEAR EQUATIONS IN $L^1(\mathbb{R})$

Stephen D. Fisher

In [2] M. G. Crandall and L. C. Evans show that the singular semi-linear problem

$$(*)\quad \begin{cases} -u''(x) + \beta(u(x)) = f(x), & -\infty < x < \infty \\ u'(\pm\infty) = 0 & \\ u'' \in L^1(\mathbb{R}) & \end{cases}$$

has a solution for each $f \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} f > 0$ if (and only if) β is integrable at $-\infty$. Here β is a given positive monotone increasing continuous function on $\mathbb{R}$. In fact, they discuss the more general situation when β is a maximal monotone graph. In this paper we consider several extensions of the problem (*) and provide another technique for proving that these equations have a solution. In particular, we recover the result of Crandall and Evans by different means.

Theorem 1. Let g be a positive continuous function on $\mathbb{R}$ with

$$\lim_{t \to -\infty} g(t) = 0, \quad \int_{-\infty}^{\infty} g(s)ds \quad \text{divergent} .$$

Let $L^1_+ = \{f \in L^1(\mathbb{R}) : \int_{\mathbb{R}} f > 0\}$; for $f \in L^1_+$ consider the problem

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$$
(1) \qquad \begin{cases} -u''(x) + g(u(x)) = f(x), & -\infty < x < \infty \\ u'' \in L^1(\mathbb{R}) \\ u'(\pm\infty) = 0 \ . \end{cases}
$$

<u>The following are equivalent:</u>

(a) (1) <u>has a solution for all</u> $f \in L^1_+$

(b) (1) <u>has a solution for some</u> $f \in L^1_+$

(c) g <u>is integrable at</u> $-\infty$.

Proof. (a) implies (b) is trivial. To see that (b) implies (c) suppose there is a function u with $u'' \in L^1$, $u'(\pm\infty) = 0$, and

$$
(2) \qquad -u'' + g(u) = f
$$

for some $f \in L^1$. Then $u' \in L^\infty$ and u tends to $-\infty$ at both $\pm\infty$ for the following reason. Suppose there is a sequence $x_n \to \infty$ with $\lim_n u(x_n) = L > -\infty$. Let $\{y_n\}$ be any other sequence of real numbers tending to $+\infty$. Then from (2) we get

$$
-\frac{1}{2}(u'(y_n))^2 + \frac{1}{2}(u'(x_n))^2 + H(u(y_n)) - H(u(x_n)) = \int_{x_n}^{y_n} fu'
$$

where

$$
H(t) = \int_0^t g(s)ds \ .
$$

Hence, $\lim_n H(u(y_n))$ exists and equals $H(u(L))$. Thus, $H(u(t))$ has a

limit at ∞ which implies that u has limit L at ∞ since H is strictly monotone. But then $g(u(t))$ tends to $g(L) > 0$ as $t \to \infty$ which contradicts the fact that $g(u(t))$ is in $L^1(\mathbb{R})$. An identical argument shows u tends to $-\infty$ at $-\infty$. With H as above we also have

$$\frac{1}{2}[(u'(y))^2 - (u'(0))^2] + H(u(0)) - H(u(y)) = \int_y^0 fu'$$

for each $y, y < 0$. Thus, $H(u(y))$ has a finite limit as $y \to -\infty$. Since $u(y) \to -\infty$ as $y \to -\infty$ we find that $H(s)$ has a finite limit as $s \to -\infty$ implying that g is integrable at $-\infty$. The proof that (c) implies (a) is the most difficult. The first step is to show that the set of those $f \in L^1_+$ for which (1) is solvable is closed in L^1_+; the second step is then obviously to show that the set of those $f \in L^1_+$ for which (1) is solvable is dense in L^1_+. To prove the first assertion, let $f_n \to f$ in $L^1(\mathbb{R})$, with f, $f_n \in L^1_+$. Let u_n satisfy

(3a) $$-u_n'' + g(u_n) = f_n$$

(3b) $$u_n'(\pm\infty) = 0$$

(3c) $$u_n'' \in L^1(\mathbb{R}) .$$

Integrate both sides of (3a) from $-\infty$ to x and then from x to $+\infty$ and use the fact that $g \geq 0$. This gives $|u_n'(x)| \leq \|f\|_1 + 1$ for all large n and hence

(4) $$\|u_n'\|_{L^\infty(\mathbb{R})} \leq A, \quad n = 1, 2, \ldots .$$

This in turn implies that $\{u_n\}$ is equicontinuous. We may assume, therefore, that $\{u_n\}$ converges uniformly on compact subsets of $\mathbb{R}$ to either $+\infty$, or $-\infty$, or to a continuous function u. Set

$$G(x) = \int_{-\infty}^{x} g(s)ds .$$

For any $x \in \mathbb{R}$ and any n we have

$$\begin{aligned} G(u_n(x)) &= \int_{-\infty}^{x} g(u_n(t))u_n'(t)dt \\ &= \frac{1}{2}[u_n'(x)]^2 + \int_{-\infty}^{x} f_n u_n' \\ &\le A_1 . \end{aligned}$$

Hence, $u_n(x) \le C$ for all n and all x. Thus, it is obviously impossible that $\{u_n\}$ tends to $+\infty$. Suppose that $\{u_n\}$ tends to $-\infty$ uniformly on compact subsets of $\mathbb{R}$. Again we have

$$-\frac{1}{2}(u_n'(x))^2 + G(u_n(x)) = \int_{-\infty}^{x} f_n(t)u_n'(t)dt \tag{5}$$

and hence

$$0 = \int_{-\infty}^{\infty} f_n u_n' . \tag{6}$$

We may assume that $\{u_n'\}$ converges weak-* in $L^\infty(\mathbb{R})$ to a function p and also that $\{u_n'(0)\}$ converges. Integrating (3a) from 0 to x we see that $u_n'(x)$ converges pointwise to $p(x)$ on $\mathbb{R}$. Hence, (5) and (6) yield

$$-\frac{1}{2}(p(x))^2 = \int_{-\infty}^{x} fp$$

and

$$0 = \int_{-\infty}^{\infty} fp.$$

Hence, p has a limit of 0 at both $+\infty$ and $-\infty$. Again from (3a) we obtain

$$u_n'(y) - u_n'(x) + \int_y^x g(u_n(t))dt = \int_y^x f_n(t)dt$$

so that

$$p(y) - p(x) = \int_y^x f(t)dt .$$

Now let $y \to -\infty$ and $x \to +\infty$; we find

$$0 < \int_{-\infty}^{\infty} f = p(-\infty) - p(+\infty) = 0,$$

a contradiction. Note that this argument is dependent on g in only a minor way. In particular, if $\{g_n\}$ is a sequence of positive continuous functions converging uniformly on compact subsets to a positive continuous function g which tends to 0 at $-\infty$ and which lies in $L^1(-\infty, 0]$ but not in $L^1(\mathbb{R})$ and if, say, $\{g_n\}$ increases to g on $(-\infty, \infty)$, then the functions v_n which satisfy

$$-v_n'' + g_n(v_n) = f, \; v_n'(\pm\infty) = 0, \; f \in L_+^1$$

are equicontinuous and uniformly bounded on compact subsets of $\mathbb{R}$. We shall make use of this later on.

Returning to the functions $\{f_n\}$ and $\{u_n\}$ we see that $\{u_n\}$ converges uniformly on compact subsets of $\mathbb{R}$ to a continuous function u. We clearly have $u_n'' \to u''$ in L_{loc}^1 so that u satisfies

(7) $$-u'' + g(u) = f \text{ on } \mathbb{R} .$$

Fatou's lemma implies $g(u)$ is in $L^1(\mathbb{R})$ and hence $u'' \in L^1(\mathbb{R})$; thus u' has limits at both $\pm\infty$ and u tends to $-\infty$ at both $\pm\infty$ as in the implication (b) implies (c). From (5) and (6) we get

$$-\frac{1}{2}(u'(x))^2 + G(u(x)) = \int_{-\infty}^{x} fu'$$

and

$$0 = \int_{-\infty}^{\infty} fu' .$$

Hence, u' tends to 0 at both $\pm\infty$, so that u is a solution of (1). Note also that

$$\int_{-\infty}^{\infty} |u'' + f| = \int_{-\infty}^{\infty} (u'' + f) = \int_{-\infty}^{\infty} f$$

$$\leq \|f\|_1$$

and hence

(8) $$\|u''\|_1 \leq 2\|f\|_1 .$$

The second assertion, that there is a dense set of $f \in L^1_+$ for which (1) is solvable, will be proved in the following way. Let f be a continuous function on $\mathbb{R}$ in L^1_+ with support in the interval $I = [a, b]$. We shall show (1) is solvable for this f. We assume temporarily that g is C^1 on $\mathbb{R}$.

We shall need the following Proposition.

Proposition. Let $a < b$ and let g be a positive C^1 function on $\mathbb{R}$ which is integrable at $-\infty$ and bounded at $+\infty$; set

$$G(x) = \int_{-\infty}^{x} g(s)ds .$$

Then for each α, β the initial value problem

$$(9) \qquad \begin{cases} -v''(x) + g(v(x)) = f(x), & a < x < b, \quad f \in L^2(a,b) \\ v(a) = \alpha, \quad v'(a) = \beta \end{cases}$$

has a unique solution. If $\alpha_n \to \alpha$ and $\beta_n \to \beta$ and if v_n is the solution of (9) for (α_n, β_n), then v_n converges uniformly to the solution v of (9) for (α, β). Finally, the family $\{v_{\alpha\beta}\}$ of solutions of (9) corresponding to the initial values $\{(\alpha, \beta) : -\infty < \alpha \leq \alpha_0, \ |\beta| \leq M\}$ is equicontinuous on $[a, b]$.

Proof. Once the equicontinuity is established the existence and uniqueness follow from standard results; see [1], Chapter 1. To obtain the equicontinuity assertion (from which the second assertion also follows), we multiply the top equation in (9) by v' and integrate to obtain

$$-\frac{1}{2}(v'(x))^2 + G(v(x)) + \frac{1}{2}\beta^2 - G(\alpha) = \int_a^x fv'$$

so that if x_0 is chosen with $|v'(x_0)| = \|v'\|_\infty$ we have

$$\begin{aligned} \|v'\|_\infty^2 &\leq \beta^2 + 2G(\alpha) + 2G(v(x_0)) + A\|v'\|_\infty \\ &\leq \beta^2 + 2G(\alpha) + 2G(\alpha + (b-a)\|v'\|_\infty) + A\|v'\|_\infty \\ &\leq \beta^2 + 2G(\alpha) + A_0 + A_1(\alpha + (b-a)\|v'\|_\infty) + A\|v'\|_\infty \end{aligned}$$

for some constants A_0, A_1 depending only on g. Hence, $\|v'\|_\infty$ is bounded for $|\beta| \leq M$ and $-\infty < \alpha \leq \alpha_0$.

Conclusion of proof of Theorem 1. Let f be a continuous function in L^1_+ with support in the interval (a, b). We shall show that (1) is solvable for this f. First, on $(-\infty, a]$ we show that the equation

$$g(u(x)) = u''(x) \tag{10}$$

$$u(a) = c_1, \; u'(-\infty) = 0$$

has a solution. Let v be the function with

$$v'(t) = (2G(t))^{-1/2}, \quad -\infty < t < c_1$$

$$v(c_1) = a$$

where

$$G(x) = \int_{-\infty}^{x} g(s)ds.$$

Then v is increasing and has range $(-\infty, a]$. Let u be the inverse of v on $(-\infty, a]$, $u(v(t)) = t$. Thus

$$u(a) = c_1$$

and

$$u'(x) = 1/v'(t) = (2G(t))^{1/2}$$

or

$$u'(x) = (2G(u(x)))^{1/2}. \tag{11}$$

If we differentiate both sides of (11) we see that u satisfies (10). Similarly, there is a solution of

$$u''(x) = g(u(x)) \qquad b < x < \infty$$

$$u(b) = c_2, \quad u'(\infty) = 0$$

which satisfies

$$u'(x) = -(2G(u(x)))^{1/2}, \quad b < x < \infty .$$

Hence, to finish the proof of the theorem we need only show that there is a solution v of the equation

$$-v'' + g(v) = f \text{ on } (a, b) \tag{12}$$

with

$$\text{(13)} \quad \begin{aligned} &\text{(a)} \qquad v'(a) = (2G(v(a)))^{1/2} \\ &\text{(b)} \qquad v'(b) = -(2G(v(b)))^{1/2} . \end{aligned}$$

Let v_t be the solution of (12) with $v(a) = t$ and $v'(a) = (2G(t))^{1/2}$ assured by the Proposition. (We temporarily assume that g is bounded at $+\infty$ if, in fact, it is not.) Then

$$\begin{aligned} v_t'(b) &= v_t'(a) + \int_a^b v_t''(s)ds \\ &= (2G(t))^{1/2} + \int_a^b g(v_t(s))ds - \rho \end{aligned}$$

where $\rho = \int_a^b f(t)dt > 0$. To show that t may be chosen with $v_t'(b) = -(2G(v_t(b)))^{1/2}$ we consider

$$\ell(t) = (2G(t))^{1/2} + (2G(v_t(b)))^{1/2} + \int_a^b g(v_t(s))ds - \rho .$$

The Proposition implies ℓ is continuous. We have

$$\ell(t) \geq -\rho + (2G(t))^{1/2} .$$

Since G is unbounded, there are values of t with $\ell(t) > 0$. Next let $t \downarrow -\infty$; by the equicontinuity of the functions $\{v_t\}$ we must have $v_t \to -\infty$ uniformly on $[a, b]$ so that $\ell(t) \to -\rho < 0$; hence, there is a t_0 at which $\ell(t_0) = 0$, and thus (12) is solvable with the boundary conditions (13).

We have now shown that (1) is solvable for all $f \in L^1_+$ under the assumption

(14) $$g \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \; g \notin L^1(\mathbb{R}) .$$

If g is merely positive and continuous on $\mathbb{R}$ with $g \in L^1(-\infty, 0)$, $g \notin L^1(\mathbb{R})$, then there is a sequence $\{g_n\}$ of positive functions satisfying (14) which converge uniformly on compact subsets of $\mathbb{R}$ to g and which also increase to g on $(-\infty, \infty)$. The comments made earlier show that the solutions $\{u_n\}$ of (1) with g_n in place of g converge to a solution of (1) for g. This completes the proof of Theorem 1.

Remark. The condition $g \notin L^1(\mathbb{R})$ is necessary as well as sufficient in order that Theorem 1 be valid. For suppose $g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$; then the

function G is bounded. If f is supported on $[-1,1]$ and if (1) has a solution for f, then (13) must hold with u in place of v so that

$$0 = \sqrt{2G(u(-1))} + \sqrt{2G(u(1))} + \int_{-1}^{1} g(u(s))ds - \int_{-1}^{1} f(s)ds .$$

The first three terms of this expression are bounded, independent of u, and hence the integral of f over $\mathbb{R}$ can not exceed some fixed number depending only on g.

Theorem 2. <u>Let</u> g <u>be a positive continuous function on</u> $\mathbb{R}$ <u>with</u>

$$\lim_{t\to-\infty} g(t) = 0, \quad g \notin L^1(\mathbb{R}) . \tag{15}$$

<u>Let</u> $B(x)$ <u>be a positive absolutely continuous function on</u> $\mathbb{R}$ <u>with</u> $B' \in L^1(\mathbb{R})$ <u>and</u> B <u>bounded away from zero</u>. <u>For</u> $f \in L^1_+$ <u>consider the equation</u>

$$\begin{cases} -u''(x) + B(x)g(u(x)) = f(x), & -\infty < x < \infty \\ u'' \in L^1(\mathbb{R}) & \\ u'(\pm\infty) = 0 . & \end{cases} \tag{16}$$

<u>Then</u> (16) <u>has a solution for each</u> $f \in L^1_+$ <u>if and only if</u> g <u>is integrable at</u> $-\infty$.

Proof. If (16) is solvable for some $f \in L^1_+$ with support in $[-1,1]$ then $u' > 0$ on $(-\infty,-1]$ and $u' < 0$ on $[1,\infty)$. It now follows very much as in Theorem 1 that u tends to $-\infty$ at $\pm\infty$ and that g is integrable at $-\infty$.

To show the sufficiency of the condition that g be integrable at $-\infty$ we first show that the equations

$$
(17) \qquad \begin{cases} u''(x) = B(x)g(u(x)), & |x| \geq a > 0 \\ u(-a) = c_1, \ u(a) = c_2 \\ u'(\pm\infty) = 0 \end{cases}
$$

have a solution. As in the proof of Theorem 1, the solution u must be monotone increasing for $-\infty < x < -a$ and monotone decreasing on (a, ∞); we shall only consider the details for the case $-\infty < x < -a$, the other case being entirely similar. We wish to find a continuous function v with

$$
(18) \qquad v'(t) = \left(2 \int_{-\infty}^{t} B(v(s))g(s)ds\right)^{-1/2}, \quad -\infty < t < c_1
$$

$$
v(c_1) = -a \ .
$$

If such a v exists, then the inverse function u of v will satisfy

$$
u'(x) = \left(2 \int_{-\infty}^{x} B(r)g(u(r))u'(r)dr\right)^{1/2}
$$

$$
u(-a) = c_1
$$

and hence u will satisfy (17). To see that (18) has a solution let b_1 and b_2 be positive numbers with $b_1 \leq B(s) \leq b_2$ for all s and let ξ_N be the function defined by

$$
\xi_N(t) = \left(2 \int_{-N}^{t} g(s)ds\right)^{-1/2}, \quad -N \leq t \leq c_1 \ .
$$

Let $\Omega_N = \{w \in C(-N, c_1) : (2b_2)^{-1/2}\xi_N(t) \le w(t) \le (2b_1)^{-1/2}\xi_N(t)$ for all $t \in [-N, c_1]\}$ and let T map Ω_N into Ω_N by

$$(Tw)(x) = \left(2\int_{-N}^{x} B(\tilde{w}(s))g(s)ds\right)^{-1/2}$$

where

$$\tilde{w}'(t) = w(t), \quad \tilde{w}(c_1) = -a .$$

Clearly $Tw \in \Omega_N$; if $\{w_n\}$ is a bounded sequence in Ω_N, then $\{\tilde{w}_N\}$ is equicontinuous and uniformly bounded. Thus, T is a compact mapping *and so has a fixed point* w_N *which must satisfy*

$$w_N(x) = \left(2\int_{-N}^{x} B(\tilde{w}_N(s))g(s)ds\right)^{-1/2}, \quad -N \le x \le c_1 .$$

The functions $\{\tilde{w}_N\}$ are equicontinuous and uniformly bounded on compact subsets of $(-\infty, c_1]$ and so a subsequence, again denoted by $\{\tilde{w}_N\}$, converges uniformly on compact subsets of $(-\infty, c_1]$ to a function $\tilde{w}_0$. But we also see that

$$\int_{-N}^{x} B(\tilde{w}_N(s))g(s)ds \to \int_{-\infty}^{x} B(\tilde{w}_0(s))g(s)ds$$

uniformly on compact subsets of $(-\infty, c_1]$. Hence, $w_N \to w_0$ uniformly on compacta; setting $v = \tilde{w}_0$ we see that v satisfies (18).

The remainder of the proof of Theorem 2 is like that of Theorem 1; the condition that $B' \in L^1(\mathbb{R})$ is used to prove that the sequence $\{u_n\}$ can not go to $-\infty$.

Corollary 3. Let $a(x) \in L^1(\mathbb{R})$, $f \in L^1(\mathbb{R})$, and let g be a positive continuous function satisfying (15). Consider the equation

$$(19)\quad \begin{cases} \text{(i)} & -u''(x) + a(x)u'(x) + g(u(x)) = f(x), \quad -\infty < x < \infty \\ \text{(ii)} & u'' \in L^1(\mathbb{R}) \\ \text{(iii)} & u'(\pm\infty) = 0 \end{cases}$$

<u>Let</u> g <u>be integrable at</u> $-\infty$ <u>and set</u> $w(x) = \exp[-\int_0^x a(s)ds]$. <u>A necessary and sufficient condition that</u> (19) <u>be solvable is that</u>

$$(20)\quad \int_{\mathbb{R}} f(x)w(x)dx > 0 .$$

<u>If</u> (21) <u>is solvable for all</u> $f \in L^1$ <u>satisfying</u> (20), <u>then</u> g <u>is integrable at</u> $-\infty$.

Proof. Let $x = H(y)$ where H is the inverse of the function I defined by

$$I'(x) = 1/w(x)$$

$$I(0) = 0 .$$

Then both H and I are 1-1 monotone increasing functions mapping $\mathbb{R}$ onto $\mathbb{R}$ and the substitution $v(y) = u(H(y))$ reduces (19) to

$$(21)\quad \begin{cases} -v''(y) + (H'(y))^2 g(v(y)) = (H'(y))^2 f(H(y)) \\ v'' \in L^1, \quad v'(\pm\infty) = 0 \end{cases}$$

which has a solution according to Theorem 2 precisely when

$$0 < \int_{-\infty}^{\infty} (H'(y))^2 f(H(y))dy$$

$$= \int_{-\infty}^{\infty} f(x)w(x)dx .$$

Remark. Let β be a maximal monotone graph lying in the upper half-plane; that is, $\beta(x)$ is a subset of $\{y > 0\}$ for each $x \in \mathbb{R}$. Let $\beta^{o}(x) = \min\{y : y \in \beta(x)\}$. The result of Crandall and Evans is that if

$$\int_{-\infty}^{a} \beta^{o}(x)dx < \infty$$

for some $a \in D(\beta)$, then the equation

$$(22) \qquad -u''(x) + \beta(u(x)) \ni f(x), \quad u'(\pm\infty) = 0, \quad f \in L^{1}_{+}$$

is solvable. This result also follows from Theorem 1 in the following way. Let $\{\beta_n\}$ be a sequence of positive continuous monotone increasing functions which increase to β^{o} on $D(\beta)$ and which increase to $+\infty$ off $D(\beta)$. The solutions $\{u_n\}$ of (1) with β_n in place of g then decrease on $\mathbb{R}$ to a solution u of (22).

A final result related to Theorem 1 is presented below.

Theorem 4. <u>Let</u> g <u>be a positive continuous function on</u> $\mathbb{R}$ <u>satisfying</u> (15). <u>For</u> $f \in L^{1}(\mathbb{R})$ <u>consider the equation</u>

$$(23) \qquad \begin{cases} u''(x) + g(u(x)) = f(x), & -\infty < x < \infty \\ u'' \in L^{1}(\mathbb{R}) & \\ u'(-\infty) = \xi_1, \ u'(+\infty) = \xi_2 & \end{cases}$$

where

$$(24) \qquad \int_{-\infty}^{\infty} f(x)dx = \rho > \xi_2 - \xi_1 .$$

(a) <u>Suppose</u> g <u>is integrable at</u> $-\infty$. <u>If</u> (23) <u>has a solution for some</u> f <u>with compact support (which necessarily satisfies</u> (24)) <u>then</u> $\xi_1 > 0 > \xi_2$. <u>If</u> (23) <u>has a solution for</u> $f \equiv 0$, <u>then</u> $\xi_1 = -\xi_2$.

(b) <u>If</u> g <u>is integrable at</u> $-\infty$ <u>and if</u> $\xi_1 > 0 > \xi_2$, <u>then</u> (23) <u>has a solution for all</u> f <u>with</u>

$$\xi_2 - \xi_1 < \rho = \int_{-\infty}^{\infty} f(x)dx \leq \min\{\xi_2, -\xi_1\} . \tag{25}$$

(c) <u>If</u> (23) <u>has a solution for some</u> f <u>satisfying</u> (24), <u>then</u> g <u>is integrable at</u> $-\infty$.

Proof. (a). If f has support in $[a,b]$, then $u''(x) < 0$ for $x < a$ and $x > b$. If $u'(-\infty) \leq 0$, then $u' < 0$ on $(-\infty, a)$ and hence u is decreasing on $(-\infty, a)$. However, u must tend to $-\infty$ at both $-\infty$ and $+\infty$ if u is a solution of (23) and thus u can not decrease on $(-\infty, a)$. Likewise, $u'(+\infty)$ must be negative. Further, if $u'' + g(u) \equiv 0$, then

$$(u'(x))^2 + 2G(u(x)) \equiv \text{const. on } (-\infty, \infty)$$

which clearly implies that $\xi_1 = -\xi_2$.

(c) is proved exactly as in Theorem 1.

(b) is the most difficult of the assertions. First, exactly as in Theorem 1, it can be shown that the set of those f satisfying (24) for which (23) is solvable is closed in $L^1(\mathbb{R})$. Next, we show that if f has compact support, say in (a, b), and if f satisfies

$$\xi_2 - \xi_1 < \rho = \int_{-\infty}^{\infty} f(x)dx < \min(-\xi_1, \xi_2) \tag{25'}$$

then (23) has a solution. The key to this, as in Theorem 1, is to show two things: first that the equations

$$(26)\qquad \begin{cases} u''(x) + g(u(x)) = 0, & x \notin [a,b] \\ u'(-\infty) = \xi_1,\ u'(\infty) = \xi_2 \end{cases}$$

have a solution which necessarily satisfies

$$(27)\qquad \begin{aligned} u'(a) &= (\xi_1^2 - 2G(u(a)))^{1/2} \\ u'(b) &= -(\xi_2^2 - 2G(u(b)))^{1/2} \end{aligned}$$

and second that the equation

$$(28)\qquad u''(x) + g(u(x)) = f(x), \qquad a \le x \le b$$

is solvable subject to the non-linear boundary conditions (27). Both these assertions are proved as the similar statements are in the proof of Theorem 1.

Remark. The upper bound in (25) is not completely satisfactory; however, the situation for (23) is more involved than that of (1) as (a) shows.

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Let g be a positive continuous function on $\mathbb{R}$ which tends to zero at $-\infty$ and which is not integrable over $\mathbb{R}$. The boundary-value problem $-u'' + g(u) = f$, $u'(\pm\infty) = 0$, is considered for $f \in L^1(\mathbb{R})$. ~~We show that~~ this problem can have a solution if and only if g is integrable at $-\infty$ and if this is so then the problem is solvable precisely when $\int_{-\infty}^{\infty} f(t)dt > 0$.* Some extensions of this result are also given.

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